# Sum and Product of Quadratic Roots Worksheet and Solutions

**Q1**

Write down the sum and the product of the roots for each of the following equations:

**(a)** $2x^2 + 12x - 3 = 0$, **(b)** $x^2 - 8x + 5 = 0$.

***Solution***

**(a)** $2x^2 + 12x - 3 = 0$

Here $a = 2$, $b = 12$ and $c = -3$.

Take careful note of the signs.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{12}{2} = -6$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-3}{2} = -1\frac{1}{2}$.

**(b)** $x^2 - 8x + 5 = 0$

Here $a = 1$, $b = -8$ and $c = 5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-8}{1} = 8$.

Notice the double negative.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$.

**Q2**

Find the sum and the product of the roots of each of the following quadratic equations:

**(a)** $4x^2 + 8x = 5$, **(b)** $x(x - 4) = 6 - 2x$.

***Solution***

Neither equation is in the form $ax^2 + bx + c = 0$ and so the first thing to do is to get them into this standard form.

**(a)** $4x^2 + 8x = 5$

$\Rightarrow 4x^2 + 8x - 5 = 0$

Now in the form $ax^2 + bx + c = 0$.

In this case $a = 4$, $b = 8$ and $c = -5$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{8}{4} = -2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-5}{4} = -\frac{5}{4}$.

**(b)** $x(x - 4) = 6 - 2x$

Expand the brackets and take everything onto the LHS.

$\Rightarrow x^2 - 4x + 2x - 6 = 0$

$\Rightarrow x^2 - 2x - 6 = 0$

Now in the standard form.

Here $a = 1$, $b = -2$ and $c = -6$.

The sum of roots, $\alpha + \beta = -\frac{b}{a} = -\frac{-2}{1} = 2$.

The product of roots, $\alpha\beta = \frac{c}{a} = \frac{-6}{1} = -6$.

**Q3** Write down equations with integer coefficients for which:

**(a)** sum of roots $= 4$, product of roots $= -7$,

**(b)** sum of roots $= -4$, product of roots $= 15$,

**(c)** sum of roots $= -\frac{3}{5}$, product of roots $= -\frac{1}{2}$.

## Solution

**(a)** A quadratic equation can be written as

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - (4)x + (-7) = 0$

$\Rightarrow \quad x^2 - 4x - 7 = 0$

**(b)** Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - (-4)x + (15) = 0$

$\Rightarrow \quad x^2 + 4x + 15 = 0$

Again great care must be taken with the signs.

**(c)** Using

$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

gives $x^2 - (-\frac{3}{5})x + (-\frac{1}{2}) = 0$

$\Rightarrow \quad x^2 + \frac{3}{5}x - \frac{1}{2} = 0$

$\Rightarrow \quad 10x^2 + 10(\frac{3}{5})x - 10(\frac{1}{2}) = 0$

$\Rightarrow \quad 10x^2 + 6x - 5 = 0$

Some of the coefficients are fractions not integers. You can eliminate the fractions by multiplying throughout by 10 or $(2 \times 5)$.

You now have integer coefficients.

**Q4** Given that $\alpha + \beta = 4$ and $\alpha\beta = 7$, find the values of:

**(a)** $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$, **(b)** $\alpha^2\beta^2$.

## Solution

**(a)** You need to write this expression in terms of $\alpha + \beta$ and $\alpha\beta$ in order to use the values given in the question.

Now $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta + \alpha}{\alpha\beta} = \dfrac{\alpha + \beta}{\alpha\beta}$

$\Rightarrow \quad \dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{4}{7}$

**(b)** $\alpha^2\beta^2 = (\alpha\beta)^2$

$\Rightarrow \quad \alpha^2\beta^2 = 7^2 = 49$

Two relations which will prove very useful are

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

Notice how the expressions on the RHS contain combinations of just $\alpha + \beta$ and $\alpha\beta$.

These two results can be proved fairly easily:

$(\alpha + \beta)^2 = (\alpha + \beta)(\alpha + \beta) \Rightarrow (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

as required

and

$(\alpha + \beta)^3 = (\alpha + \beta)(\alpha + \beta)(\alpha + \beta)$

$\Rightarrow \quad (\alpha + \beta)^3 = (\alpha + \beta)(\alpha^2 + 2\alpha\beta + \beta^2)$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 2\alpha^2\beta + \alpha\beta^2 + \alpha^2\beta + 2\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$

$\Rightarrow \quad (\alpha + \beta)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha + \beta)$

Take $3\alpha\beta$ as a factor.

$\Rightarrow \quad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$ as required

| Q5 | Given that $\alpha + \beta = 5$ and $\alpha\beta = -2$, find the values of: **(a)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, **(b)** $\alpha^3 + \beta^3$, **(c)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. |
|---|---|

## Solution

**(a)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

Now it is in terms of $\alpha + \beta$ and $\alpha\beta$.

Substitute the known values of $\alpha + \beta$ and $\alpha\beta$

$\Rightarrow \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{5^2 - 2(-2)}{-2} = \frac{25 + 4}{-2} = -\frac{29}{2}$

**(b)** $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$\Rightarrow \quad \alpha 3 + \beta^3 = 5^3 - 3(-2)(5) = 125 + 30 = 155$

**(c)** $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$\Rightarrow \quad \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{5^2 - 2(-2)}{(-2)^2} = \frac{25 + 4}{4} = \frac{29}{4}$

| Q6 | Given that $\alpha + \beta = 5$ and $\alpha\beta = \frac{2}{3}$, find the value of $(\alpha - \beta)^2$. |
|---|---|

## Solution

$$(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\Rightarrow \quad (\alpha - \beta)^2 = 5^2 - 4\left(\tfrac{2}{3}\right) = 25 - \tfrac{8}{3} = 22\tfrac{1}{3}$$

## Worked example 1.7

Write the expression $\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2}$ in terms of $\alpha + \beta$ and $\alpha\beta$.

## Solution

$$\frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{\alpha^3 + \beta^3}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)}{(\alpha\beta)^2}$$

## Section 2

**1** Find the sums and products of the roots of the following equations:

**(a)** $x^2 - 6x + 4 = 0$, **(b)** $2x^2 + x - 5 = 0$.

**2** Write down the equations, with integer coefficients, where:

**(a)** sum of roots $= -4$, product of roots $= 7$,

**(b)** sum of roots $= \frac{5}{4}$, product of roots $= -\frac{1}{2}$.

**3** The roots of the equation $x^2 + 5x - 6 = 0$ are $\alpha$ and $\beta$. Find the values of:

**(a)** $\alpha^2 + \beta^2$ **(b)** $\frac{\beta}{\alpha} + \frac{\alpha}{\beta}$.

**4** Given that the roots of $3x^2 - 9x + 1 = 0$ are $\alpha$ and $\beta$, find a quadratic equation whose roots are $\frac{1}{\alpha\beta^2}$ and $\frac{1}{\alpha^2\beta}$.

## Solutions

**1** **(a)** sum $= 6$, product $= 4$;

**(b)** sum $= -\frac{1}{2}$, product $= -\frac{5}{2}$.

**2** **(a)** $x^2 + 4x + 7 = 0$; **(b)** $4x^2 - 5x - 2 = 0$.

**3** **(a)** 37; **(b)** $-\frac{37}{6}$.

**4** $x^2 - 27x + 27 = 0$.